

Quantum κ -Poincaré Algebra from de Sitter Space of Momenta

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Abstract

There is a growing number of physical models, like point particle(s) in 2+1 gravity or Doubly Special Relativity, in which the space of momenta is curved, de Sitter space. We show that for such models the algebra of space-time symmetries possesses a natural Hopf algebra structure. It turns out that this algebra is just the quantum κ -Poincaré algebra.

1 Introduction

It was observed some time ago by Majid [1] that in phase space curvature and non-commutativity play dual role. It is well known, of course, that for curved space-time momenta do not commute. And vice versa, for more than 50 years, since the seminal work of Snyder [2], we know that curved momentum space implies non-commutative space-time. The importance of this duality was appreciated even more in the recent years, when the deep relation between (at least some types of) space-time non-commutativity and Hopf algebra structure of space-time symmetries has been understood. It has been shown that there exists another duality between non-commutativity of one space in the pair and non-triviality of the co-product of another space (e.g., in the case of κ -Poincaré algebra the non-triviality of co-product of momenta leads by Heisenberg double procedure [3] to non-commutativity of positions.)

In particular it turned out that these duality makes it possible to construct phase space of Doubly Special Relativity (DSR). DSR has been formulated [4], [5] as a generalization of special relativity describing the kinematics of particles at energies close to Planck scale. This theory possess two observer-independent scales, of velocity and mass (identified with velocity of light and Planck mass.) It soon turned out [6], [7] that the formal structure of DSR can be based on κ -Poincaré algebra [8], and as the result of nontrivial Hopf structure of this algebra the space time of DSR must be non-commutative [9], [10] (for reviews of recent developments in DSR see [11].)

The κ -Poincaré algebra is a deformed algebra of space-time symmetries, preserving the momentum scale κ . It reads³

$$[M_i, M_j] = i \epsilon_{ijk} M_k, \quad [M_i, N_j] = i \epsilon_{ijk} N_k,$$

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³In another version of DSR theory proposed by Magueijo and Smolin [12] the algebra is different, but it can be shown to be related to the one here by change of variables [13], [14]. In particular the space-time non-commutativity is the same as in (3).

$$[N_i, N_j] = -i \epsilon_{ijk} M_k. \quad (1)$$

$$[M_i, p_j] = i \epsilon_{ijk} p_k, \quad [M_i, p_0] = 0$$

$$[N_i, p_j] = i \delta_{ij} \left(\frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) + \frac{\mathbf{p}^2}{2\kappa} \right) - i \frac{1}{\kappa} p_i p_j, \quad [N_i, p_0] = i p_i. \quad (2)$$

It follows [9], [10], [14] that the coordinates on dual κ -Minkowski space-time form the algebra

$$[x_0, x_i] = -\frac{i}{\kappa} x_i \quad (3)$$

as a result of a non-trivial co-product of momenta

$$\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0 \quad (4)$$

$$\Delta(p_i) = p_i \otimes 1 + e^{-p_0/\kappa} \otimes p_i \quad (5)$$

In the parallel development it was shown in [15], [16] that one can interpret DSR theory as a theory with curved momentum space, being de Sitter space (it turns out that Snyder space-time is also a particular instance of DSR.) Then the positions and Lorentz transformation generators algebra can be understood as the $\mathfrak{so}(4, 1)$ algebra of vectors tangent to the origin of de Sitter space.

It is interesting to note that there exist a model, in which curved space of momenta was not put by hand in one or another way, but instead derived from first principles. This model is gravity in 2+1 dimensions coupled to point particle(s) (the clear and detailed exposition can be found in [17], where the reader can also find references to the earlier papers.) It was realized recently that 2+1 gravity coupled to point particles is an example of DSR theory; the claim has been put forward that similarly in 3+1 dimensions DSR can be understood as a flat space limit of (quantum) gravity coupled to point particles; see [18], [19] for details.

In this paper we clarify the issue of relation between curved space of momenta and Hopf algebra structure of space-time symmetries. They both lead to the non-commutative structure of κ -Minkowski space-time, but the relation between them was not clear. Here we show that in the case of de Sitter space of momenta, the algebra of space-time symmetries acquires naturally the structure of Hopf algebra, and the resulting Hopf algebra is κ -Poincaré algebra, as expected.

2 Iwasawa decomposition of $\mathfrak{so}(1, 4)$ algebra

Suppose we have a theory, in which the space of momenta is four dimensional de Sitter space. As explain in Introduction we have to do with such a setting in DSR (and in the case of 2+1 gravity coupled to a point particle, in which case de Sitter space is 3 dimensional.) The group of symmetries of this space is $SO(1, 4)$ with the algebra $\mathfrak{so}(1, 4)$. The elements of this algebra correspond to infinitesimal Lorentz transformations, which form the subalgebra $\mathfrak{so}(1, 3)$, and infinitesimal translations of momenta, which can be identified with positions.

We can uniquely decompose the algebra $\mathfrak{so}(1, 4)$ into a direct sum of subalgebras (so called Iwasawa decomposition, see e.g., [20])

$$\mathfrak{so}(1, 4) = \hat{\mathfrak{k}} + \hat{\mathfrak{n}} + \hat{\mathfrak{a}} \quad (6)$$

where algebra $\hat{\mathfrak{k}}$ is the $\mathfrak{so}(1, 3)$ algebra, $\hat{\mathfrak{a}}$ is generated by the element

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7)$$

and the algebra $\hat{\mathfrak{n}}$ by the elements

$$\mathbf{n}_i = \begin{bmatrix} 0 & \epsilon_i & 0 \\ (\epsilon_i)^T & 0 & -(\epsilon_i)^T \\ 0 & \epsilon_i & 0 \end{bmatrix} \quad (8)$$

where ϵ_i are versors in i 'th direction ($\epsilon_1 = (1, 0, 0)$, etc), and T denotes transposition. Every element belonging to algebra $\hat{\mathfrak{n}}$ is a positive root of element H, whose value is equal one, i.e.,

$$[H, \hat{\mathfrak{n}}] = \hat{\mathfrak{n}} \quad (9)$$

Note the similarity of this algebra and the κ -Minkowski space-time algebra (3). It is easy to see that (3) can be obtained from (9) by identification $x_0 = -\frac{i}{\kappa} H$, $x_i = -\frac{i}{\kappa} \mathbf{n}_i$.

It follows that every element g of the $SO(1, 4)$ group can be decomposed as follows

$$g = (kna) \quad \text{or} \quad g = (k \vartheta na)$$

Here $k \in SO(1, 3)$, the element a belongs to group A generated by H

$$A = \exp\left(-\frac{p_0}{\kappa} H\right) = \begin{bmatrix} \cosh \frac{p_0}{\kappa} & 0 & 0 & 0 & -\sinh \frac{p_0}{\kappa} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\sinh \frac{p_0}{\kappa} & 0 & 0 & 0 & \cosh \frac{p_0}{\kappa} \end{bmatrix} \quad (10)$$

and the element n belongs to group N generated by algebra $\hat{\mathfrak{n}} = p_i \mathbf{n}_i$

$$N = \exp\left(\frac{1}{\kappa} \hat{n}\right) = \begin{bmatrix} 1 + \frac{1}{2\kappa^2} \vec{p}^2 & \frac{p_1}{\kappa} & \frac{p_2}{\kappa} & \frac{p_3}{\kappa} & -\frac{1}{2\kappa^2} \vec{p}^2 \\ \frac{p_1}{\kappa} & 0 & 0 & 0 & -\frac{p_1}{\kappa} \\ \frac{p_2}{\kappa} & 0 & 0 & 0 & -\frac{p_2}{\kappa} \\ \frac{p_3}{\kappa} & 0 & 0 & 0 & -\frac{p_3}{\kappa} \\ \frac{1}{2\kappa^2} \vec{p}^2 & \frac{p_1}{\kappa} & \frac{p_2}{\kappa} & \frac{p_3}{\kappa} & 1 - \frac{1}{2\kappa^2} \vec{p}^2 \end{bmatrix} \quad (11)$$

and $\vartheta = \text{diag}(-1, 1, 1, 1, -1)$. Note that, as a result of isomorphism between the algebra $\hat{\mathfrak{n}} + \hat{\mathfrak{a}}$ (9) and the algebra (3) the element of NA can be equivalently expressed as an ordered plane wave on κ -Minkowski space-time

$$\exp(ip_i x_i) \exp(-ip_0 x_0)$$

The decomposition of group $\text{SO}(1,4)$ described above is unique. The coset space $\text{SO}(1,4)/\text{SO}(1,3)$ is de Sitter space and acting with the subgroup NA on the stability point \mathcal{O}

$$\mathcal{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \kappa \end{bmatrix} \quad (12)$$

we get the following coordinate system

$$\begin{aligned} \eta_0 &= -\kappa \sinh \frac{p_0}{\kappa} - \frac{\vec{p}^2}{2\kappa} e^{\frac{p_0}{\kappa}} \\ \eta_i &= -p_i e^{\frac{p_0}{\kappa}} \\ \eta_4 &= \kappa \cosh \frac{p_0}{\kappa} - \frac{\vec{p}^2}{2\kappa} e^{\frac{p_0}{\kappa}} \end{aligned} \quad (13)$$

on de Sitter space, being the four dimensional hypersurface

$$-\eta_0^2 + \eta_i^2 + \eta_4^2 = \kappa^2$$

in five dimensional space of Lorentzian signature.

It is easily seen that the coordinates p_0, p_i describe only half of de Sitter space and therefore the group KNA is not the whole group $\text{SO}(1,4)$. However if we define

$$\begin{aligned} \eta_0 &= \mp \left(\kappa \sinh \frac{p_0}{\kappa} - \frac{\vec{p}^2}{2\kappa} e^{\frac{p_0}{\kappa}} \right) \\ \eta_i &= \mp p_i e^{\frac{p_0}{\kappa}} \\ \eta_4 &= \pm \left(\kappa \cosh \frac{p_0}{\kappa} - \frac{\vec{p}^2}{2\kappa} e^{\frac{p_0}{\kappa}} \right) \end{aligned} \quad (14)$$

then the group $\text{SO}(1,4)$ can be written in the form

$$\text{SO}(1,4) = \text{KNA} \cup \text{K}\mathcal{I}\text{NA} \quad (15)$$

(these two parts are disjoint.) This decomposition generalizes the Iwasawa decomposition.

Thus we have the following picture: the subalgebra $\hat{\mathbf{n}} + \hat{\mathbf{a}}$ defines κ -Minkowski space-time, while energy and momenta are just functions on the group NA generated by the algebra $\hat{\mathbf{n}} + \hat{\mathbf{a}}$.

3 Hopf algebra structure of momenta

In this section we show that from the group structure we can deduce the coalgebra structure of momentum Hopf algebra. Following the general scheme we compute co-product, antipode and counit for momenta p_μ , which can be interpreted as functions on the group NA.

Let G be a group and let $\mathcal{A} = \text{Fun } G$ be complex associative algebra of functions on G with a unit element. The multiplication $(f_1, f_2) \rightarrow f_1 f_2$ and the unit I in \mathcal{A} are defined by the formula

$$(f_1, f_2)(g) = f_1(g)f_2(g), \quad I(g) \equiv 1 \quad (16)$$

The multiplication is a mapping $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. The algebra \mathcal{A} is commutative. The group operations allow us to introduce other operations on \mathcal{A} , namely (see, e.g., [20])

1. the comultiplication $\triangle : \mathcal{A} \equiv \text{Fun } G \rightarrow \text{Fun } (G \times G)$,
2. the counit $\varepsilon : \mathcal{A} \rightarrow \mathbb{C}$,
3. the antipode $S : \mathcal{A} \rightarrow \mathcal{A}$.

defined by the formulas

$$\begin{aligned} (\triangle f)(g_1, g_2) &= f(g_1 g_2), \quad g_1 g_2 \in G, \\ \varepsilon(f) &= f(e), \\ (Sf)(g) &= f(g^{-1}), \quad g \in G. \end{aligned} \quad (17)$$

The group properties lead to some properties of homomorphisms \triangle , ε and S , to wit

$$(\triangle \otimes id) \circ \triangle = (id \otimes \triangle) \circ \triangle, \quad (\varepsilon \otimes id) \circ \triangle = (id \otimes \varepsilon) \circ \triangle = id \quad (18)$$

$$(m \circ (S \otimes id) \circ \triangle)(f) = (m \circ (id \otimes S) \circ \triangle)(f) = \varepsilon(f)I(g) \quad (19)$$

It follows that algebra of functions on a group $\mathcal{A} \equiv \text{Fun}(G)$ is a Hopf algebra.

Now we apply this formalism to the group NA defined above. The idea is to take as the algebra of functions \mathcal{A} on NA, the momenta p_0 and p_i . As we will see, in this way we can equip the space of momenta with Hopf algebra structure. Using matrix representation of the group NA one can easily deduce the following property

$$\exp(-ip_0 x_0) \exp(ip_i x_i) = \exp(ip_i e^{-p_0/\kappa} x_i) \exp(-ip_0 x_0) \quad (20)$$

Using this property for the product of two group elements we get

$$\begin{aligned} &\exp(ip_i^{(1)} x_i) \exp(-ip_0^{(1)} x_0) \exp(ip_i^{(2)} x_i) \exp(-ip_0^{(2)} x_0) \\ &\exp i \left(p_i^{(1)} + e^{-p_0^{(1)}/\kappa} p_i^{(2)} \right) x_i \exp -i(p_0^{(1)} + p_0^{(2)}) x_0 \end{aligned} \quad (21)$$

while for the inverse we have

$$(\exp(ip_i x_i) \exp(-ip_0 x_0))^{-1} = \exp(-ip_i x_i e^{p_0/\kappa}) \exp(ip_0 x_0) \quad (22)$$

Comparing these formulas with the definitions (17) we easily find that

$$\triangle(p_i) = p_i \otimes \mathbb{1} + e^{-p_0/\kappa} \otimes p_i, \quad (23)$$

$$\Delta(p_0) = p_0 \otimes \mathbb{1} + \mathbb{1} \otimes p_0, \quad (24)$$

$$\varepsilon(p_0) = \varepsilon(p_i) = 0, \quad (25)$$

$$S(p_0) = -p_0, \quad S(p_i) = -p_i e^{p_0/\kappa} \quad (26)$$

We see therefore that on the algebra of momenta of DSR, being functions on de Sitter space, one can introduce the structure of Hopf algebra, and this structure is the same as in the κ -Poincaré algebra (4), (5). Let us now turn to the remaining part of κ -Poincaré, the deformed algebra of Lorentz symmetries.

4 Coalgebra structure of $U(\mathfrak{so}(1, 3))$ algebra

The coalgebra structure of $\mathfrak{so}(1, 3)$ algebra can be deduced from the group action. From Iwasawa decomposition described above we know that any element of the group $SO(1, 4)$ can be uniquely decomposed in two ways

$$K_1 N_1 A_1 = N_2 A_2 K_2$$

which is equivalent to

$$K_1 N_1 A_1 K_2^{-1} = N_2 A_2$$

Since this decomposition is unique the above equation defines the action of $SO(1, 3)$ group on group NA. We can write explicitly

$$K_1 e^{ip_i x_i} e^{-ip_0 x_0} K_2^{-1} = e^{ip'_i x_i} e^{-ip'_0 x_0} \quad (27)$$

where $K_1, K_2 \in SO(1, 3)$ and k_2 depends on p_i, p_0 and K_1 . Now if take $K_1 = 1$ infinitesimal equation (27) implies

$$K_1 e^{ip_i x_i} e^{-ip_0 x_0} = \delta(e^{ip_i x_i} e^{-ip_0 x_0}) + e^{ip_i x_i} e^{-ip_0 x_0} K_2 \quad (28)$$

If we take $K_1 = 1 + i\xi N_i$ we get

$$K_2 = 1 + i\xi(e^{-p_0/\kappa} N_i + \frac{1}{\kappa} \epsilon_{ijk} p_j M_k) \quad (29)$$

and

$$\begin{aligned} & \delta(e^{ip_i x_i} e^{-ip_0 x_0}) = \\ & = e^{ip_i x_i} i\xi \left(-p_i x_0 + \left(\delta_{ij} \left(\frac{\kappa}{2} (1 - e^{-2p_0/\kappa}) + \frac{\mathbf{p}^2}{2\kappa} \right) - \frac{1}{\kappa} p_i p_j \right) x_j \right) e^{-ip_0 x_0}. \end{aligned} \quad (30)$$

Using above formulas we can write equation (28) in the following form

$$(1 + i\xi N_i) e^{ip_i x_i} e^{-ip_0 x_0} (1 - i\xi(e^{-p_0/\kappa} N_i + \frac{1}{\kappa} \epsilon_{ijk} p_j M_k)) \approx e^{ip'_i x_i} e^{-ip'_0 x_0} \quad (31)$$

where we remember that this equation is exact only up to linear terms in ξ . Variables p'_i, p'_0 have the following form

$$p'_j = 1 - i\xi[N_i, p_j], \quad p'_0 = 1 - i\xi[N_i, p_0] \quad (32)$$

where commutators are defined in equation (2).

Using notation from preceding section we can write

$$(1 + i\xi N_i).f(g) = f((1 + i\xi N_i)gK_2^{-1}) \quad (33)$$

where $g \in \text{NA}$, f is a function on the group and dot means action by commutator.

The antipode $S(N_i)$ can be defined from the action on inverse elements, to wit

$$(1 + i\xi N_i)(e^{ip_i x_i} e^{-ip_0 x_0})^{-1} K_2^{-1} = e^{ip'_i x_i} e^{-ip'_0 x_0}. \quad (34)$$

From the uniqueness of Iwasawa decomposition it appears that there is only one set p'_0, p'_i, K_2 satisfying above equation. We define the antipode $S(N_i)$ in the following way

$$K_2^{-1} = 1 - i\xi S(N_i) \quad (35)$$

Using notation from preceding section and definition (33) we can write the left hand side of equation (34) in the form

$$(1 + i\xi N_i).f(g^{-1}) = f\left((1 + i\xi S(N_i))g(1 - i\xi N_i)^{-1}\right) \quad (36)$$

which is equivalent to

$$(1 + i\xi N_i).(Sf)(g) = (S(1 + i\xi S(N_i)).f)(g) \quad (37)$$

From this we can easily compute the explicit form of the antipode $S(N_i)$

$$S(N_i) = -e^{\frac{p_0}{\kappa}}(N_i - \frac{1}{\kappa}\epsilon_{ijk}p_j M_k) \quad (38)$$

In order to derive the coproduct for $\mathfrak{so}(1, 3)$ algebra we must rewrite the definition of coproduct in more convenient form

$$(\triangle f)(g_1, g_2) = \sum_i f_i^{(1)} \otimes f_i^{(2)}(g_1, g_2) = f(g_1 g_2). \quad (39)$$

where we understand above formula as multiplication

$$\sum_i f_i^{(1)} \otimes f_i^{(2)}(g_1, g_2) = \sum_i f_i^{(1)}(g_1) f_i^{(2)}(g_2)$$

We derive the coproduct of N_i from group action on product of elements belonging to group NA. This means that we can act either on the product of translations (i.e., on NA) or first on translations and then take the product. We have

$$f(K_1 g_1 g_2 K_2^{-1}) = f(K_1 g_1 h^{-1} h g_2 K_2^{-1}) = \sum_i f_i^{(1)} \otimes f_i^{(2)}(K_1 g_1 h^{-1}, h g_2 K_2^{-1}), \quad (40)$$

and for $K = 1 + i\xi N_i$ we get

$$(1 + i\xi N_i).f(g_1 g_2) = (1 + i\xi \sum_j h_j^{(1)} \otimes h_j^{(2)}) \cdot \sum_i f_i^{(1)} \otimes f_i^{(2)}(g_1, g_2) \quad (41)$$

where

$$\Delta(N_i) = \sum_j h_j^{(1)} \otimes h_j^{(2)}$$

is the Sweedler notation for coproduct. One calculates

$$\Delta(N_i) = N_i \otimes 1 + e^{-p_0/\kappa} \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} p_j \otimes M_k. \quad (42)$$

For the counit $\varepsilon(N_i)$ we take

$$\varepsilon(N_i) \cdot f(g) = N_i \cdot (\varepsilon f)(g) \Rightarrow \varepsilon(N_i) = 0 \quad (43)$$

This whole procedure can be repeated for generators of rotation M_i , and the result is

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i, \quad S(M_i) = -M_i, \quad \varepsilon(M_i) = 0. \quad (44)$$

From the group properties one can deduce the corresponding properties for antipode counit and coproduct (see (18)), which are necessary to define Hopf algebra.

5 Conclusions

In this paper we show that when the phase space has the space of momenta of the form of de Sitter space, in this space one can quite naturally introduce the structure of Hopf algebra. This algebra can be further extended to κ -Poincaré algebra of all ten space-time symmetries.

This result can be rather easily extended to the case when the momentum part of the phase space is anti-de Sitter space (as in the case of 2+1 gravity, as described in [17] and DSR with space-like deformation [21]), and, most likely, to the case of arbitrary symmetric space. The fact that in these cases the algebra of space-time symmetries has the structure of Hopf algebra may have profound physical consequences, which should be closely investigated.

It should be also noted that it follows from the investigations presented here, that similar, though, in a sense, upside-down structure arises in theories on de Sitter space-time (though in this case there would be a nontrivial Hopf algebra of positions and Lorentz transformations.) It is not unlikely that this observation may shed some new light on quantum field theory on de Sitter space, which is a basis of inflationary cosmology.

References

- [1] S. Majid, *Foundation of Quantum Group Theory*, ch. 6, CUP, 1995.
- [2] H. S. Snyder, "Quantized Space-Time," *Phys. Rev.* **71** (1947) 38.
- [3] J. Lukierski and A. Nowicki, *Proceedings of Quantum Group Symposium at Group 21*, (July 1996, Goslar) Eds. H.-D. Doebner and V.K. Dobrev, Heron Press, Sofia, 1997, p. 186.

- [4] G. Amelino-Camelia, “Testable scenario for relativity with minimum-length,” *Phys. Lett. B* **510**, 255 (2001) [arXiv:hep-th/0012238].
- [5] G. Amelino-Camelia, “Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale,” *Int. J. Mod. Phys. D* **11**, 35 (2002) [arXiv:gr-qc/0012051].
- [6] J. Kowalski-Glikman, “Observer independent quantum of mass,” *Phys. Lett. A* **286** (2001) 391 [arXiv:hep-th/0102098].
- [7] N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, “Deformed boost transformations that saturate at the Planck scale,” *Phys. Lett. B* **522** (2001) 133 [arXiv:hep-th/0107039].
- [8] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, “Q deformation of Poincare algebra,” *Phys. Lett. B* **264** (1991) 331.
- [9] S. Majid and H. Ruegg, “Bicrossproduct structure of kappa Poincare group and noncommutative geometry,” *Phys. Lett. B* **334** (1994) 348 [arXiv:hep-th/9405107].
- [10] J. Lukierski, H. Ruegg and W. J. Zakrzewski, “Classical quantum mechanics of free kappa relativistic systems,” *Annals Phys.* **243** (1995) 90 [arXiv:hep-th/9312153].
- [11] G. Amelino-Camelia, “Some encouraging and some cautionary remarks on doubly special relativity in quantum gravity,” arXiv:gr-qc/0402092; J. Kowalski-Glikman, “Introduction to doubly special relativity,” arXiv:hep-th/0405273, *Lecture Notes in Physics*, to appear.
- [12] J. Magueijo and L. Smolin, “Lorentz invariance with an invariant energy scale,” *Phys. Rev. Lett.* **88** (2002) 190403 [arXiv:hep-th/0112090].
- [13] J. Kowalski-Glikman and S. Nowak, “Doubly special relativity theories as different bases of kappa-Poincare algebra,” *Phys. Lett. B* **539** (2002) 126 [arXiv:hep-th/0203040].
- [14] J. Kowalski-Glikman and S. Nowak, “Non-commutative space-time of doubly special relativity theories,” *Int. J. Mod. Phys. D* **12** (2003) 299 [arXiv:hep-th/0204245].
- [15] J. Kowalski-Glikman, “De Sitter space as an arena for doubly special relativity,” *Phys. Lett. B* **547** (2002) 291 [arXiv:hep-th/0207279].
- [16] J. Kowalski-Glikman and S. Nowak, “Doubly special relativity and de Sitter space,” *Class. Quant. Grav.* **20** (2003) 4799 [arXiv:hep-th/0304101].
- [17] H. J. Matschull and M. Welling, “Quantum mechanics of a point particle in 2+1 dimensional gravity,” *Class. Quant. Grav.* **15** (1998) 2981 [arXiv:gr-qc/9708054].

- [18] G. Amelino-Camelia, L. Smolin and A. Starodubtsev, “Quantum symmetry, the cosmological constant and Planck scale phenomenology,” *Class. Quant. Grav.* **21** (2004) 3095 [arXiv:hep-th/0306134].
- [19] L. Freidel, J. Kowalski-Glikman and L. Smolin, “2+1 gravity and doubly special relativity,” *Phys. Rev. D* **69** (2004) 044001 [arXiv:hep-th/0307085].
- [20] N. Ja. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions*, Kluwer 1992.
- [21] A. Blaut, M. Daszkiewicz, J. Kowalski-Glikman and S. Nowak, “Phase spaces of doubly special relativity,” *Phys. Lett. B* **582** (2004) 82 [arXiv:hep-th/0312045].